

Test 7

Section I, Part A

Allotted Time: 62 Minutes

Number of Questions: 29 Questions

Calculator Utilization: The use of a calculator is not permitted for this part of the test.

Instructions: Work through each of the problems, using the space provided to jot down calculations or notes. Carefully examine the answer choices for each question and determine which is the most accurate. Once you've selected the best answer, fill in the matching oval on the answer sheet. Keep in mind that anything written in this booklet won't be graded. Don't get stuck on any single question for too long. In this test, unless stated otherwise, assume that the domain of a function f includes all real values of x that result in $f(x)$ being a real number:

1. Evaluate

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}.$$

- a. 1
 - b. 2
 - c. 3
 - d. The limit does not exist
2. If

$$f(x) = 4x^4 + 5e^x - 6 \cos x,$$

then the derivative is.

- a. $16x^3 + 5e^x - 6 \sin x$
- b. $16x^3 + 5e^x + 6 \sin x$
- c. $4x^3 + 5e^x + 6 \sin x$
- d. $16x^3 - 5e^x + 6 \sin x$

3. The derivative of a function f is

$$f'(x) = (x - 1)(x + 2)^2.$$

Which statement is true?

- a. f has no local extremum at $x = -2$ and a local minimum at $x = 1$.
 - b. f has a local maximum at $x = -2$ and a local minimum at $x = 1$.
 - c. f has local minima at both $x = -2$ and $x = 1$.
 - d. f has a local minimum at $x = -2$ and no local extremum at $x = 1$.
4. Let

$$g(x) = (\sin x + 2)^4.$$

Find $g'(x)$.

- a. $4(\sin x + 2)^3$
 - b. $4(\sin x + 2)^3 \sin x$
 - c. $(\sin x + 2)^4 \cos x$
 - d. $4(\sin x + 2)^3 \cos x$
5. Find

$$\int (6x^2 - 3e^x + 4 \sin x) dx.$$

- a. $12x - 3e^x - 4 \cos x + C$
 - b. $2x^3 - 3e^x + 4 \cos x + C$
 - c. $2x^3 + 3e^x - 4 \cos x + C$
 - d. $2x^3 - 3e^x - 4 \cos x + C$
6. Let

$$f(x) = \begin{cases} kx - 2, & x < 3, \\ x^2 + 1, & x \geq 3. \end{cases}$$

What value of k makes f continuous at $x = 3$?

- a. 2
- b. $\frac{8}{3}$
- c. 3
- d. 4

7. A particle moves along a line with velocity

$$v(t) = 2t^2 - 3t + 1$$

for $0 \leq t \leq 2$. What is the displacement of the particle from $t = 0$ to $t = 2$?

- a. $-\frac{2}{3}$
- b. 0
- c. 1
- d. $\frac{4}{3}$

8. If

$$h(x) = x^2 e^{-x},$$

then $h'(x) =$

- a. $2xe^{-x}$
- b. $e^{-x}(x^2 + 2x)$
- c. $e^{-x}(2x - x^2)$
- d. $e^{-x}(x^2 - 2x)$

9. Suppose

$$f''(x) = (x - 2)^2(x + 1).$$

Which statement is true?

- a. f is concave up on $(-\infty, -1)$ and concave down on $(-1, \infty)$.
 - b. f has points of inflection at $x = -1$ and $x = 2$.
 - c. f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.
 - d. f is concave down on $(-\infty, -1)$ and concave up on $(-1, 2) \cup (2, \infty)$, with an inflection point at $x = -1$ only.
10. Find the total area between

$$y = x^2 - 9$$

and the x -axis on the interval $[-4, 4]$.

- a. 36
- b. $\frac{116}{3}$
- c. $\frac{124}{3}$
- d. $\frac{128}{3}$

11. The graph of f consists of line segments from $(0, 0)$ to $(2, 4)$, from $(2, 4)$ to $(5, 4)$, from $(5, 4)$ to $(7, 0)$, and from $(7, 0)$ to $(9, -2)$. What is

$$\int_0^9 f(x) \, dx?$$

- a. 14
 - b. 16
 - c. 18
 - d. 20
12. The curve is defined by

$$x^2y + y^2 = 12.$$

What is $\frac{dy}{dx}$ at the point $(2, 2)$?

- a. -2
 - b. -1
 - c. 1
 - d. 2
13. Let

$$f(x) = x^2 + 4x$$

on the interval $[-1, 3]$. What value of c satisfies the conclusion of the Mean Value Theorem?

- a. 1
 - b. 2
 - c. 3
 - d. 4
14. Evaluate

$$\lim_{x \rightarrow 2} \left(\frac{x+1}{(x-2)^2} \right).$$

- a. 0
- b. $\frac{3}{4}$
- c. The limit does not exist because the one-sided limits are different signs
- d. ∞

15. Let

$$G(x) = \int_1^{\sin x} (t^2 + 1) dt.$$

What is $G'(x)$?

- a. $\sin^2 x + 1$
 - b. $(x^2 + 1) \cos x$
 - c. $(\sin^2 x + 1) \cos x$
 - d. $\cos^2 x + 1$
16. The side length of a square is increasing at a rate of 5 centimeters per second. How fast is the area of the square increasing when the side length is 6 centimeters?
- a. $30 \text{ cm}^2/\text{s}$
 - b. $45 \text{ cm}^2/\text{s}$
 - c. $60 \text{ cm}^2/\text{s}$
 - d. $90 \text{ cm}^2/\text{s}$
17. For the differential equation

$$\frac{dy}{dx} = (x + 2)(y - 1),$$

at which point is the slope of the solution curve equal to 0?

- a. $(0, 0)$
 - b. $(3, 1)$
 - c. $(-1, 2)$
 - d. $(2, 4)$
18. If

$$h(x) = \frac{x^2 + 1}{e^x},$$

then $h'(x)$ is

- a. $\frac{2x}{e^x}$
- b. $e^x(2x - x^2 - 1)$
- c. $e^{-x}(2x - x^2 - 1)$
- d. $e^{-x}(x^2 + 2x + 1)$

19. Evaluate

$$\int_0^1 8x(2x^2 + 3)^3 dx.$$

- a. 272
- b. 136
- c. 544
- d. $\frac{625}{2}$

20. Let

$$f(x) = x^3 - 3x^2 - 9x + 5$$

on the interval $[-2, 4]$. What is the absolute minimum value of f on the interval?

- a. -15
- b. 3
- c. -22
- d. 10

21. Suppose $f(4) = -1$ and $f'(4) = 5$. What is $(f^{-1})'(-1)$?

- a. 5
- b. $\frac{1}{5}$
- c. $-\frac{1}{5}$
- d. -5

22. Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin(5x)}{2x}.$$

- a. 0
- b. 1
- c. $\frac{5}{2}$
- d. 5

23. Which definite integral is represented by

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(2 + \frac{3i}{n}\right)^2 \frac{3}{n}?$$

- a. $\int_2^5 x^2 dx$
- b. $\int_0^3 (2 + x)^2 dx$
- c. $\int_0^2 (3 + x)^2 dx$
- d. $\int_2^3 x^2 dx$

24. The graph of f' has the following sign behavior:

- $f'(x) > 0$ for $x < -2$
- $f'(-2) = 0$
- $f'(x) > 0$ for $-2 < x < 1$
- $f'(1) = 0$
- $f'(x) < 0$ for $1 < x < 4$
- $f'(4) = 0$
- $f'(x) > 0$ for $x > 4$

Which statement is true?

- f has a local maximum at $x = -2$ and local minima at $x = 1$ and $x = 4$.
- f has local minima at $x = -2$ and $x = 4$, and a local maximum at $x = 1$.
- f has no local extrema.
- f has no local extremum at $x = -2$, a local maximum at $x = 1$, and a local minimum at $x = 4$.

25. A particle moves along a line with velocity

$$v(t) = t^2 - 7t + 10$$

and acceleration

$$a(t) = 2t - 7$$

for $0 < t < 6$. On which interval(s) is the particle speeding up?

- $(0, 2)$ only
- $(2, \frac{7}{2})$ and $(5, 6)$
- $(\frac{7}{2}, 5)$ only
- $(0, 2)$ and $(\frac{7}{2}, 5)$

26. Which function satisfies

$$\frac{dy}{dx} = 3xy$$

and the initial condition $y(0) = 4$?

- $y = 4e^{3x}$
- $y = e^{3x^2/2} + 3$
- $y = 4 + e^{3x^2/2}$
- $y = 4e^{3x^2/2}$

27. The region bounded by $y = \sqrt{x+1}$, $y = 0$, $x = 0$, and $x = 3$ is revolved about the x -axis. What is the volume of the resulting solid?

- a. 6π
- b. 7π
- c. $\frac{13\pi}{2}$
- d. $\frac{15\pi}{2}$

28. The limit

$$\lim_{h \rightarrow 0} \frac{(2+h)^4 - 2^4}{h}$$

is equal to

- a. 32
 - b. 16
 - c. 8
 - d. 4
29. A rectangle has one side on the x -axis and its upper two vertices on the parabola

$$y = 9 - x^2.$$

If the rectangle is symmetric about the y -axis, what is the maximum possible area of the rectangle?

- a. 12
- b. 18
- c. $24\sqrt{2}$
- d. $12\sqrt{3}$

Section I, Part B

Allotted Time: 38 Minutes

Number of Questions: 13 Questions

Calculator Utilization: The use of a calculator is permitted for the questions in this part of the test.

Instructions: For each problem below, solve using the space provided for scratch work. Review the format of the answer options, select the best choice, and mark the corresponding oval on your answer sheet. Note that any work written in the test booklet will not receive credit, so focus on completing the answer sheet. Avoid spending excessive time on any single question. The exact numerical answer may not always be included in the answer choices. In such cases, select the option that most closely approximates the exact value. Unless stated otherwise, the domain of a function is assumed to include all real numbers x for which $f(x)$ produces a real number.

30. Values of a continuous function f are shown in the table below.

x	0	1	3	4.5	6	8
$f(x)$	12	18	21	19	15	10

Using the trapezoidal rule with the intervals shown in the table, approximate

$$\int_0^8 f(x) dx.$$

- a. 128.0
- b. 134.5
- c. 141.0
- d. 148.5

31. The curve is defined implicitly by

$$x^2y + y^3 = \frac{25}{2}.$$

At the point $(1.5, 2)$, approximate $\frac{dy}{dx}$.

- a. -0.74
- b. -0.42
- c. 0.42
- d. 0.74

32. Let

$$g(x) = xe^{-0.4x} + 0.5 \sin(2x)$$

on the interval $[0, 5]$. Use a calculator to determine the absolute maximum value of g on $[0, 5]$.

- a. 1.32
- b. 1.68
- c. 2.04
- d. 2.51

33. A cyclist's distance from home, $s(t)$, in miles, is shown in the table below.

t	2.8	2.9	3.0	3.1	3.2
$s(t)$	14.06	14.82	15.63	16.49	17.40

Which of the following is the best estimate of $s'(3)$?

- a. 7.60
- b. 8.10
- c. 8.35
- d. 9.10

34. Values of a function h near $x = 2$ are shown below.

x	1.98	1.998	1.9998	2.0002	2.002	2.02
$h(x)$	2.040	2.004	2.0004	1.9996	1.996	1.960

Based on the table, which statement is most accurate?

- a. $\lim_{x \rightarrow 2} h(x) = 0$
- b. $\lim_{x \rightarrow 2} h(x) = 2$
- c. $\lim_{x \rightarrow 2} h(x) = 4$
- d. $\lim_{x \rightarrow 2} h(x)$ does not exist

35. Let

$$A(x) = \int_1^x \frac{\sqrt{1+t^4}}{t+2} dt.$$

Use a calculator to approximate $A(4)$.

- a. 4.37
- b. 5.12
- c. 6.48
- d. 7.03

36. The region enclosed by

$$y = e^{-0.3x} + 0.1x$$

and

$$y = 0.8$$

has two intersection points. Use a calculator to find the area of the region.

- a. 0.350
- b. 0.725
- c. 1.180
- d. 1.640

37. A function y satisfies

$$\frac{dy}{dx} = \frac{x}{y+2}, \quad y(1) = 0.$$

Find $y(4)$. Round to the nearest hundredth.

- a. 2.36
- b. 2.74
- c. 3.12
- d. 3.58

38. Let

$$f(x) = e^{-0.5x} \cos(x^2).$$

Use a calculator to approximate $f'(1.2)$.

- a. -1.91
- b. -1.34
- c. -0.68
- d. 0.42

39. A twice-differentiable function f has

$$f''(x) = \cos x - 0.15x.$$

Use a calculator to approximate the x -value where f has a point of inflection on $[0,4]$.

- a. 1.365
 - b. 1.892
 - c. 2.417
 - d. 3.006
40. The output of a machine, in hundreds of units, t hours after it is turned on is modeled by

$$P(t) = 30(1 - e^{-0.4t}) - 1.2t$$

for $0 \leq t \leq 10$. Use a calculator to determine when the output is maximized.

- a. 3.214 hours
 - b. 4.486 hours
 - c. 5.756 hours
 - d. 7.118 hours
41. Find the average value of

$$f(x) = \sqrt{1 + \sin(x^2)}$$

on the interval $[0, 2]$.

- a. 0.84
 - b. 0.96
 - c. 1.05
 - d. 1.17
42. The region bounded by

$$y = 2e^{-0.2x}, \quad y = 0.3x + 0.5,$$

and the y -axis is revolved about the x -axis. Use a calculator to approximate the volume of the resulting solid.

- a. 8.42
- b. 10.77
- c. 13.49
- d. 16.35

Section II, Part A

Allotted Time: 30 Minutes

Number of Questions: 2 Questions

Calculator Utilization: The use of a calculator is permitted for the questions in this part of the test.

Instructions: Providing your set up for these problems is essential, as partial credit may be granted. If you include decimal approximations, ensure they are accurate to three decimal places.

1. A city park opens a new splash pad. The number of visitors at the splash pad, in hundreds of people, t hours after the park opens is modeled by

$$V(t) = 55 + 6 \ln(t + 1) + 10 \sin(0.5t) - 0.35t^2$$

for $0 \leq t \leq 12$.

- a. Find $V'(t)$. Include units.
- b. Use a calculator to find all critical values of V on the interval $0 < t < 12$. Give your answer to three decimal places.
- c. Find the absolute maximum number of visitors at the splash pad on the interval $0 \leq t \leq 12$. Include units and justify your answer.
- d. The number of visitors is increasing at a decreasing rate when $V'(t) > 0$ and $V''(t) < 0$. Determine the interval(s) on $0 < t < 12$ where the number of visitors is increasing at a decreasing rate. Justify your answer.

2. At an aquarium, visitors enter a special exhibit at a rate of $E(t)$ visitors per hour, where t is the number of hours after the exhibit opens. Selected values of $E(t)$ are shown below.

t (hours)	0	0.5	1.5	3	5	6
$E(t)$ (visitors/hr)	180	245	310	270	190	160

Visitors leave the exhibit at a rate modeled by

$$L(t) = 160 + 35e^{-0.12t^2}$$

visitors per hour. At time $t = 0$, there are 75 visitors in the exhibit.

- Use the trapezoidal rule with the data above to approximate $\int_0^6 E(t) dt$.
- Use a calculator to approximate $\int_0^6 L(t) dt$. Then interpret the meaning of $\int_0^6 (E(t) - L(t)) dt$ in the context of the problem.
- Approximate the number of visitors in the exhibit at time $t = 6$.
- Find the average rate of change of the number of visitors in the exhibit over the interval $0 \leq t \leq 6$. Include units.

Section II, Part B

Allotted Time: 60 Minutes

Number of Questions: 4 Questions

Calculator Utilization: The use of a calculator is not permitted for this part of the test.

3. Let f be the piecewise-defined function

$$f(x) = \begin{cases} ax^2 + bx + 5, & x \leq 2, \\ c \ln(x - 1) + x^2, & x > 2, \end{cases}$$

where a , b , and c are constants.

- Find a relationship between a and b that makes f continuous at $x = 2$.
 - Find c in terms of a and b so that f is differentiable at $x = 2$.
 - Assuming f is continuous and differentiable at $x = 2$, write an equation of the tangent line to the graph of f at $x = 2$.
 - If $a = -1$, find the values of b and c that make f continuous and differentiable at $x = 2$.
4. Let

$$f(x) = x^4 - 4x^3 - 8x^2 + 48x + 2.$$

- Find $f'(x)$ and determine all critical values of f .
- On which intervals is f increasing? On which intervals is f decreasing? Classify each critical value as the location of a local maximum, local minimum, or neither. Justify your answer.
- Find all x -values where f has a point of inflection. Justify your answer.
- Let L be the tangent line to the graph of f at $x = 1$. Find an equation for $L(x)$. Then determine whether $L(1.2)$ is an overestimate or an underestimate of $f(1.2)$. Justify your answer.

5. Let f be a continuous function defined on $0 \leq x \leq 10$. The graph of f is made of the following pieces:

- On $0 \leq x \leq 2$, f is the line segment from $(0, -2)$ to $(2, 2)$.
- On $2 \leq x \leq 6$, f is the upper semicircle with center $(4, 2)$ and radius 2.
- On $6 \leq x \leq 8$, f is the line segment from $(6, 2)$ to $(8, -2)$.
- On $8 \leq x \leq 10$, f is the line segment from $(8, -2)$ to $(10, 0)$.

Define

$$G(x) = \int_0^x f(t) dt.$$

- Find $G(2)$.
 - Find $G'(7)$.
 - Find $G(6)$.
 - Find the absolute maximum value of G on the interval $0 \leq x \leq 10$. Justify your answer.
6. A function $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y+3}$$

with initial condition

$$f(0) = -1.$$

- Show that

$$y = -3 + \sqrt{4 + \frac{2x^3}{3}}$$

satisfies the differential equation and the initial condition.

- Find the general solution to the differential equation.
- Find the particular solution that satisfies $f(0) = -1$, and determine $f(3)$.
- Determine the interval(s) on which the particular solution is increasing. Justify your answer.

Solutions

Section I, Part A

1. C	2. B	3. A	4. D	5. D
6. D	7. D	8. C	9. D	10. D
11. C	12. B	13. A	14. D	15. C
16. C	17. B	18. C	19. A	20. C
21. B	22. C	23. A	24. D	25. B
26. D	27. D	28. A	29. D	

1. Factor the numerator:

$$x^3 - 1 = (x - 1)(x^2 + x + 1).$$

For $x \neq 1$,

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1.$$

Therefore,

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 1^2 + 1 + 1 = 3.$$

2. Differentiate term by term:

$$\frac{d}{dx}(4x^4) = 16x^3,$$

$$\frac{d}{dx}(5e^x) = 5e^x,$$

and

$$\frac{d}{dx}(-6 \cos x) = 6 \sin x.$$

Therefore,

$$f'(x) = 16x^3 + 5e^x + 6 \sin x.$$

3. The factor $(x + 2)^2$ does not change sign at $x = -2$, so $f'(x)$ does not change sign there. Therefore, there is no local extremum at $x = -2$.

The sign of $f'(x)$ changes from negative to positive at $x = 1$, so f has a local minimum at $x = 1$.

4. Use the chain rule:

$$g'(x) = 4(\sin x + 2)^3 \cdot \cos x.$$

Therefore,

$$g'(x) = 4(\sin x + 2)^3 \cos x.$$

5. Integrate term by term:

$$\begin{aligned}\int 6x^2 \, dx &= 2x^3, \\ \int -3e^x \, dx &= -3e^x,\end{aligned}$$

and

$$\int 4 \sin x \, dx = -4 \cos x.$$

Therefore,

$$\int (6x^2 - 3e^x + 4 \sin x) \, dx = 2x^3 - 3e^x - 4 \cos x + C.$$

6. Continuity at $x = 3$ requires the two pieces to agree at $x = 3$:

$$3k - 2 = 3^2 + 1.$$

So

$$3k - 2 = 10,$$

which gives

$$k = 4.$$

7. Displacement is the integral of velocity:

$$\int_0^2 (2t^2 - 3t + 1) \, dt.$$

Find the antiderivative:

$$\int (2t^2 - 3t + 1) \, dt = \frac{2t^3}{3} - \frac{3t^2}{2} + t.$$

Then

$$\left[\frac{2t^3}{3} - \frac{3t^2}{2} + t \right]_0^2 = \frac{16}{3} - 6 + 2 = \frac{4}{3}.$$

8. Use the product rule:

$$h'(x) = 2xe^{-x} + x^2(-e^{-x}).$$

Factor out e^{-x} :

$$h'(x) = e^{-x}(2x - x^2).$$

9. Since $(x - 2)^2 \geq 0$, the sign of $f''(x)$ is determined by $x + 1$, except at $x = 2$ where $f''(x) = 0$ but does not change sign.

Thus, $f''(x) < 0$ on $(-\infty, -1)$ and $f''(x) \geq 0$ on $(-1, \infty)$, so f has an inflection point only at $x = -1$.

10. The function crosses the x -axis at $x = -3$ and $x = 3$. The total area is

$$2 \int_3^4 (x^2 - 9) \, dx + \int_{-3}^3 (9 - x^2) \, dx.$$

First,

$$\int_3^4 (x^2 - 9) \, dx = \left[\frac{x^3}{3} - 9x \right]_3^4 = \frac{10}{3}.$$

So the two outside regions have total area

$$2 \cdot \frac{10}{3} = \frac{20}{3}.$$

Also,

$$\int_{-3}^3 (9 - x^2) \, dx = 36.$$

Therefore, the total area is

$$36 + \frac{20}{3} = \frac{128}{3}.$$

11. Find the signed areas of each piece. From 0 to 2, the triangle has area $\frac{1}{2}(2)(4) = 4$. From 2 to 5, the rectangle has area $(3)(4) = 12$. From 5 to 7, the triangle has area $\frac{1}{2}(2)(4) = 4$. From 7 to 9, the triangle is below the x -axis, so its signed area is $-\frac{1}{2}(2)(2) = -2$. Therefore,

$$\int_0^9 f(x) \, dx = 4 + 12 + 4 - 2 = 18.$$

12. Differentiate implicitly:

$$x^2y + y^2 = 12$$

$$2xy + x^2 \frac{dy}{dx} + 2y \frac{dy}{dx} = 0.$$

Then

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + 2y}.$$

At $(2, 2)$,

$$\frac{dy}{dx} = \frac{-2(2)(2)}{2^2 + 2(2)} = -1.$$

13. The average rate of change on $[-1, 3]$ is

$$\frac{f(3) - f(-1)}{3 - (-1)} = \frac{21 - (-3)}{4} = 6.$$

Since $f'(x) = 2x + 4$, set $f'(c) = 6$:

$$2c + 4 = 6.$$

Therefore, $c = 1$.

14. As $x \rightarrow 2$, the numerator approaches 3, while the denominator approaches 0 through positive values because $(x - 2)^2 > 0$ on both sides of 2. Therefore,

$$\lim_{x \rightarrow 2} \frac{x + 1}{(x - 2)^2} = \infty.$$

15. By the Fundamental Theorem of Calculus and the chain rule,

$$G'(x) = ((\sin x)^2 + 1) \cdot \frac{d}{dx}(\sin x).$$

Therefore,

$$G'(x) = (\sin^2 x + 1) \cos x.$$

16. The area of a square is $A = s^2$. Differentiate with respect to time:

$$\frac{dA}{dt} = 2s \frac{ds}{dt}.$$

At $s = 6$ and $\frac{ds}{dt} = 5$,

$$\frac{dA}{dt} = 2(6)(5) = 60.$$

17. The slope is zero when

$$(x + 2)(y - 1) = 0.$$

So either $x = -2$ or $y = 1$. Of the answer choices, only $(3, 1)$ has $y = 1$.

18. Rewrite the function as

$$h(x) = (x^2 + 1)e^{-x}.$$

Use the product rule:

$$h'(x) = 2xe^{-x} - (x^2 + 1)e^{-x} = e^{-x}(2x - x^2 - 1).$$

19. Let $u = 2x^2 + 3$, so $du = 4x \, dx$ and $8x \, dx = 2 \, du$. When $x = 0$, $u = 3$. When $x = 1$, $u = 5$. Therefore,

$$\int_0^1 8x(2x^2 + 3)^3 \, dx = 2 \int_3^5 u^3 \, du.$$

$$2 \left[\frac{u^4}{4} \right]_3^5 = \frac{1}{2} (5^4 - 3^4) = 272.$$

20. First find critical points:

$$f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1).$$

So the critical points are $x = -1$ and $x = 3$. Evaluate f at the critical points and endpoints:

$$f(-2) = 3, \quad f(-1) = 10, \quad f(3) = -22, \quad f(4) = -15.$$

The absolute minimum value is -22 .

21. Use the inverse derivative formula:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Since $f(4) = -1$, we have

$$(f^{-1})'(-1) = \frac{1}{f'(f^{-1}(-1))} = \frac{1}{f'(4)} = \frac{1}{5}.$$

22. Rewrite the limit as

$$\frac{\sin(5x)}{2x} = \frac{5}{2} \cdot \frac{\sin(5x)}{5x}.$$

Since $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$, the limit is $\frac{5}{2}$.

23. Here $\Delta x = \frac{3}{n}$ and $x_i = 2 + \frac{3i}{n}$. The interval starts at 2 and has length 3, so the upper bound is

5. Therefore, the sum represents

$$\int_2^5 x^2 \, dx.$$

24. At $x = -2$, f' does not change sign, so there is no local extremum. At $x = 1$, f' changes from positive to negative, so f has a local maximum. At $x = 4$, f' changes from negative to positive, so f has a local minimum.

25. A particle is speeding up when velocity and acceleration have the same sign. Since

$$v(t) = t^2 - 7t + 10 = (t - 2)(t - 5),$$

$v(t) < 0$ on $(2, 5)$ and $v(t) > 0$ on $(0, 2) \cup (5, 6)$. Also, $a(t) < 0$ on $(0, \frac{7}{2})$ and $a(t) > 0$ on $(\frac{7}{2}, 6)$. The signs match on

$$\left(2, \frac{7}{2}\right) \quad \text{and} \quad (5, 6).$$

26. Separate variables:

$$\frac{1}{y} dy = 3x dx.$$

Integrate:

$$\ln|y| = \frac{3x^2}{2} + C.$$

Thus $y = Ce^{3x^2/2}$. Since $y(0) = 4$, $C = 4$, so

$$y = 4e^{3x^2/2}.$$

27. Using disks, the radius is $\sqrt{x+1}$. Therefore,

$$V = \pi \int_0^3 (\sqrt{x+1})^2 dx = \pi \int_0^3 (x+1) dx.$$

$$V = \pi \left[\frac{x^2}{2} + x \right]_0^3 = \pi \left(\frac{9}{2} + 3 \right) = \frac{15\pi}{2}.$$

28. The limit represents the derivative of $f(x) = x^4$ at $x = 2$. Since $f'(x) = 4x^3$,

$$f'(2) = 4(2)^3 = 32.$$

29. Let the upper-right vertex of the rectangle be $(x, 9 - x^2)$, where $0 < x < 3$. Since the rectangle is symmetric about the y -axis, its width is $2x$ and its height is $9 - x^2$. Thus

$$A(x) = 2x(9 - x^2) = 18x - 2x^3.$$

Differentiate:

$$A'(x) = 18 - 6x^2.$$

Set $A'(x) = 0$:

$$18 - 6x^2 = 0 \quad \Rightarrow \quad x^2 = 3 \quad \Rightarrow \quad x = \sqrt{3}.$$

The maximum area is

$$A(\sqrt{3}) = 2\sqrt{3}(9 - 3) = 12\sqrt{3}.$$

Section I, Part B

30. B	31. B	32. A	33. C	34. B
35. A	36. A	37. A	38. B	39. A
40. C	41. D	42. C		

30. Use trapezoids on each interval:

$$\frac{1}{2}(1)(12 + 18) + \frac{1}{2}(2)(18 + 21) + \frac{1}{2}(1.5)(21 + 19) + \frac{1}{2}(1.5)(19 + 15) + \frac{1}{2}(2)(15 + 10).$$

This gives

$$15 + 39 + 30 + 25.5 + 25 = 134.5.$$

31. Differentiate implicitly:

$$x^2 \frac{dy}{dx} + 2xy + 3y^2 \frac{dy}{dx} = 0.$$

So

$$\frac{dy}{dx} = -\frac{2xy}{x^2 + 3y^2}.$$

At (1.5, 2),

$$\frac{dy}{dx} = -\frac{2(1.5)(2)}{(1.5)^2 + 3(2)^2} \approx -0.42.$$

32. Use a calculator's maximum feature on the interval $[0, 5]$, or solve $g'(x) = 0$ and check endpoints. The absolute maximum value is approximately

$$1.32.$$

33. Use a symmetric difference quotient:

$$s'(3) \approx \frac{s(3.1) - s(2.9)}{3.1 - 2.9} = \frac{16.49 - 14.82}{0.2} = 8.35.$$

34. The values from the left and right both approach 2. Therefore,

$$\lim_{x \rightarrow 2} h(x) = 2.$$

35. Use calculator integration:

$$A(4) = \int_1^4 \frac{\sqrt{1+t^4}}{t+2} dt \approx 4.37.$$

36. Use a calculator to solve

$$e^{-0.3x} + 0.1x = 0.8.$$

The intersections occur at approximately $x = 1.370$ and $x = 6.633$. The area is

$$\int_{1.370}^{6.633} (0.8 - e^{-0.3x} - 0.1x) \, dx \approx 0.350.$$

37. Separate variables:

$$(y + 2) \, dy = x \, dx.$$

Integrate:

$$\frac{y^2}{2} + 2y = \frac{x^2}{2} + C.$$

Using $y(1) = 0$ gives $C = -\frac{1}{2}$. At $x = 4$,

$$\frac{y^2}{2} + 2y = \frac{15}{2}.$$

Solving gives

$$y \approx 2.36.$$

38. Use a numerical derivative feature or a symmetric difference quotient at $x = 1.2$. This gives

$$f'(1.2) \approx -1.34.$$

39. Inflection points occur where $f''(x) = 0$ and concavity changes. Solving

$$\cos x - 0.15x = 0$$

on $[0, 4]$ gives

$$x \approx 1.365.$$

40. Use a calculator to maximize $P(t)$ on $[0, 10]$, or solve $P'(t) = 0$ and check endpoints. The maximum occurs at approximately

$$t = 5.756.$$

41. Average value is

$$\frac{1}{2} \int_0^2 \sqrt{1 + \sin(x^2)} \, dx.$$

Using calculator integration,

$$\frac{1}{2} \int_0^2 \sqrt{1 + \sin(x^2)} \, dx \approx 1.17.$$

42. First use a calculator to solve

$$2e^{-0.2x} = 0.3x + 0.5.$$

The positive intersection is approximately $x = 2.432$. Using washers,

$$V = \pi \int_0^{2.432} [(2e^{-0.2x})^2 - (0.3x + 0.5)^2] dx.$$

Calculator integration gives

$$V \approx 13.49.$$

Section II, Part A

1. (a) Differentiate:

$$V'(t) = \frac{6}{t+1} + 5 \cos(0.5t) - 0.7t$$

The units are hundreds of visitors per hour.

- (b) Critical values occur where
- $V'(t) = 0$
- .

Using a calculator on $0 < t < 12$,

$$\frac{6}{t+1} + 5 \cos(0.5t) - 0.7t = 0$$

at approximately

$$t = 2.931$$

Therefore, the only critical value on $0 < t < 12$ is $t \approx 2.931$.

- (c) Check the endpoint values and the critical value.

$$V(0) = 55$$

$$V(2.931) \approx 70.151$$

$$V(12) \approx 17.196$$

The greatest value is 70.151 hundred visitors, so the absolute maximum number of visitors is approximately

$$7,015 \text{ visitors}$$

and it occurs about 2.931 hours after the park opens.

- (d) First find the second derivative.

$$V''(t) = -\frac{6}{(t+1)^2} - 2.5 \sin(0.5t) - 0.7$$

From part (b), $V'(t) > 0$ on $(0, 2.931)$ and $V'(t) < 0$ on $(2.931, 12)$.

Using a calculator, $V''(t) = 0$ at approximately $t = 6.931$ and $t = 11.969$. Also, $V''(t) < 0$ on $(0, 6.931)$ and $(11.969, 12)$.

The number of visitors is increasing at a decreasing rate where both $V'(t) > 0$ and $V''(t) < 0$. Therefore, the interval is

$$(0, 2.931).$$

2. (a) Use trapezoids on each interval shown in the data:

$$\begin{aligned}\int_0^6 E(t) dt &\approx \frac{1}{2}(0.5)(180 + 245) + \frac{1}{2}(1)(245 + 310) + \frac{1}{2}(1.5)(310 + 270) \\ &\quad + \frac{1}{2}(2)(270 + 190) + \frac{1}{2}(1)(190 + 160) \\ &= 106.25 + 277.5 + 435 + 460 + 175 \\ &= 1453.75\end{aligned}$$

So,

$$\int_0^6 E(t) dt \approx 1453.75$$

visitors entered the exhibit during the first 6 hours.

- (b) Using a calculator,

$$\int_0^6 L(t) dt = \int_0^6 (160 + 35e^{-0.12t^2}) dt \approx 1049.247$$

The expression

$$\int_0^6 (E(t) - L(t)) dt$$

represents the net change in the number of visitors in the exhibit during the first 6 hours.

- (c) From parts (a) and (b), the net change in visitors is approximately

$$1453.75 - 1049.25 = 404.50$$

Since there were initially 75 visitors in the exhibit,

$$75 + 404.50 = 479.50$$

Therefore, there are approximately 480 visitors in the exhibit at time $t = 6$.

- (d) The average rate of change of the number of visitors in the exhibit is

$$\frac{479.50 - 75}{6 - 0} \approx 67.417$$

Equivalently, this is

$$\frac{404.50}{6} \approx 67.417$$

So the number of visitors in the exhibit increased at an average rate of approximately 67.417 visitors per hour over the first 6 hours.

Section II, Part B

3. (a) Continuity at $x = 2$ requires the left-hand value and right-hand limit to be equal.

From the left,

$$f(2) = a(2)^2 + b(2) + 5 = 4a + 2b + 5.$$

From the right, substitute $x = 2$ into the right-hand expression because it is continuous for $x > 1$:

$$c \ln(2 - 1) + 2^2 = c \ln(1) + 4 = 4.$$

Since $\ln(1) = 0$, continuity requires

$$4a + 2b + 5 = 4.$$

Therefore,

$$4a + 2b = -1.$$

- (b) Differentiability at $x = 2$ requires the left and right derivatives to match.

For $x \leq 2$,

$$f'(x) = 2ax + b,$$

so the left-hand derivative at $x = 2$ is

$$4a + b.$$

For $x > 2$,

$$f'(x) = \frac{c}{x-1} + 2x,$$

so the right-hand derivative at $x = 2$ is

$$c + 4.$$

Set the derivatives equal:

$$4a + b = c + 4.$$

Therefore,

$$c = 4a + b - 4.$$

- (c) Since f is continuous at $x = 2$, the point on the curve is $(2, 4)$. Since f is differentiable there, the slope is

$$f'(2) = 4a + b.$$

An equation of the tangent line is

$$y - 4 = (4a + b)(x - 2).$$

- (d) From part (a), continuity requires

$$4a + 2b = -1.$$

Substitute $a = -1$:

$$4(-1) + 2b = -1.$$

$$-4 + 2b = -1$$

$$2b = 3$$

$$b = \frac{3}{2}.$$

From part (b),

$$c = 4a + b - 4.$$

Substitute $a = -1$ and $b = \frac{3}{2}$:

$$c = 4(-1) + \frac{3}{2} - 4.$$

$$c = -8 + \frac{3}{2} = -\frac{13}{2}.$$

Therefore,

$$b = \frac{3}{2} \text{ and } c = -\frac{13}{2}.$$

4. (a) Differentiate:

$$f'(x) = 4x^3 - 12x^2 - 16x + 48.$$

Factor:

$$f'(x) = 4(x + 2)(x - 2)(x - 3).$$

Critical values occur where $f'(x) = 0$ or where $f'(x)$ is undefined. Since f' is defined for all real numbers, set

$$4(x + 2)(x - 2)(x - 3) = 0.$$

Therefore, the critical values are

$$x = -2, \quad x = 2, \quad x = 3.$$

(b) Use the sign of

$$f'(x) = 4(x + 2)(x - 2)(x - 3).$$

- On $(-\infty, -2)$, $f'(x) < 0$, so f is decreasing.
- On $(-2, 2)$, $f'(x) > 0$, so f is increasing.
- On $(2, 3)$, $f'(x) < 0$, so f is decreasing.
- On $(3, \infty)$, $f'(x) > 0$, so f is increasing.

Thus, f is increasing on

$$(-2, 2) \cup (3, \infty)$$

and decreasing on

$$(-\infty, -2) \cup (2, 3).$$

- At $x = -2$, f' changes from negative to positive, so f has a local minimum.
- At $x = 2$, f' changes from positive to negative, so f has a local maximum.

- At $x = 3$, f' changes from negative to positive, so f has a local minimum.

(c) Find the second derivative:

$$f''(x) = 12x^2 - 24x - 16.$$

Set $f''(x) = 0$:

$$12x^2 - 24x - 16 = 0.$$

Divide by 4:

$$3x^2 - 6x - 4 = 0.$$

Use the quadratic formula:

$$x = \frac{6 \pm \sqrt{36 + 48}}{6} = \frac{6 \pm \sqrt{84}}{6} = \frac{3 \pm \sqrt{21}}{3}.$$

Since $f''(x)$ is a quadratic that changes sign at both values, f has points of inflection at

$$x = \frac{3 - \sqrt{21}}{3} \quad \text{and} \quad x = \frac{3 + \sqrt{21}}{3}.$$

(d) First find the point and slope at $x = 1$.

$$f(1) = 1 - 4 - 8 + 48 + 2 = 39.$$

Also,

$$f'(1) = 24.$$

So the tangent line is

$$y - 39 = 24(x - 1).$$

Therefore,

$$L(x) = 24x + 15.$$

To determine whether $L(1.2)$ is an overestimate or underestimate, use concavity near $x = 1$.

$$f''(1) = 12(1)^2 - 24(1) - 16 = -28.$$

Since $f''(1) < 0$ and f is concave down near $x = 1$, the graph of f lies below its tangent line near $x = 1$. Therefore, $L(1.2)$ is an overestimate of $f(1.2)$.

5. (a) On $0 \leq x \leq 2$, the line segment from $(0, -2)$ to $(2, 2)$ has equation

$$f(x) = 2x - 2.$$

Therefore,

$$G(2) = \int_0^2 (2t - 2) dt.$$

$$G(2) = [t^2 - 2t] \Big|_0^2 = 4 - 4 = 0.$$

So,

$$G(2) = 0.$$

(b) By the Fundamental Theorem of Calculus,

$$G'(x) = f(x).$$

At $x = 7$, $f(7)$ lies on the line segment from $(6, 2)$ to $(8, -2)$. That line segment crosses the x -axis halfway between $x = 6$ and $x = 8$, so

$$f(7) = 0.$$

Therefore,

$$G'(7) = 0.$$

(c) Since

$$G(6) = \int_0^6 f(t) dt,$$

use the signed area from 0 to 6.

From part (a),

$$\int_0^2 f(t) dt = 0.$$

From $x = 2$ to $x = 6$, the graph is an upper semicircle with center $(4, 2)$ and radius 2. This region has a rectangle of area

$$4 \cdot 2 = 8$$

plus a semicircle of area

$$\frac{1}{2}\pi(2)^2 = 2\pi.$$

Therefore,

$$G(6) = 0 + 8 + 2\pi.$$

So,

$$G(6) = 8 + 2\pi.$$

(d) Since

$$G'(x) = f(x),$$

G is increasing where $f(x) > 0$ and decreasing where $f(x) < 0$.

On $0 \leq x \leq 2$, $f(x) = 2x - 2$, so $f(x) < 0$ on $(0, 1)$ and $f(x) > 0$ on $(1, 2)$.

From $x = 2$ to $x = 6$, the graph is above the x -axis, so $f(x) > 0$.

From $x = 6$ to $x = 8$, the graph crosses the x -axis at $x = 7$. Thus, $f(x) > 0$ on $(6, 7)$ and $f(x) < 0$ on $(7, 8)$.

From $x = 8$ to $x = 10$, the graph is below the x -axis until it reaches 0 at $x = 10$, so $f(x) < 0$ on $(8, 10)$.

Therefore, G increases from $x = 1$ to $x = 7$ and decreases after $x = 7$, so the absolute maximum occurs at $x = 7$.

From part (c),

$$G(6) = 8 + 2\pi.$$

From $x = 6$ to $x = 7$, the area is a triangle with base 1 and height 2:

$$\int_6^7 f(t) dt = \frac{1}{2}(1)(2) = 1.$$

Thus,

$$G(7) = 8 + 2\pi + 1 = 9 + 2\pi.$$

The absolute maximum value of G on $0 \leq x \leq 10$ is

$$9 + 2\pi.$$

6. (a) Given

$$y = -3 + \sqrt{4 + \frac{2x^3}{3}},$$

differentiate using the chain rule:

$$\frac{dy}{dx} = \frac{1}{2} \left(4 + \frac{2x^3}{3} \right)^{-1/2} \cdot 2x^2.$$

Therefore,

$$\frac{dy}{dx} = \frac{x^2}{\sqrt{4 + \frac{2x^3}{3}}}.$$

From the proposed solution,

$$y + 3 = \sqrt{4 + \frac{2x^3}{3}}.$$

So,

$$\frac{x^2}{y + 3} = \frac{x^2}{\sqrt{4 + \frac{2x^3}{3}}}.$$

Thus,

$$\frac{dy}{dx} = \frac{x^2}{y+3},$$

so the function satisfies the differential equation.

Also,

$$y(0) = -3 + \sqrt{4} = -3 + 2 = -1.$$

Therefore, the function satisfies the initial condition.

(b) Separate the variables:

$$\frac{dy}{dx} = \frac{x^2}{y+3}$$

$$(y+3) dy = x^2 dx.$$

Integrate both sides:

$$\int (y+3) dy = \int x^2 dx.$$

$$\frac{y^2}{2} + 3y = \frac{x^3}{3} + C.$$

This is a general solution to the differential equation.

(c) Use the initial condition $f(0) = -1$ in the general solution:

$$\frac{(-1)^2}{2} + 3(-1) = \frac{0^3}{3} + C.$$

$$\frac{1}{2} - 3 = C$$

$$C = -\frac{5}{2}.$$

So,

$$\frac{y^2}{2} + 3y = \frac{x^3}{3} - \frac{5}{2}.$$

To solve explicitly, multiply by 2:

$$y^2 + 6y = \frac{2x^3}{3} - 5.$$

Complete the square:

$$y^2 + 6y + 9 = \frac{2x^3}{3} + 4.$$

$$(y + 3)^2 = \frac{2x^3}{3} + 4.$$

Since $f(0) = -1$, we need $y + 3 = 2$, so choose the positive square root:

$$y = -3 + \sqrt{4 + \frac{2x^3}{3}}.$$

Therefore,

$$f(3) = -3 + \sqrt{4 + \frac{2(3)^3}{3}}.$$

$$f(3) = -3 + \sqrt{22}.$$

(d) The solution is increasing when

$$\frac{dy}{dx} > 0.$$

From the differential equation,

$$\frac{dy}{dx} = \frac{x^2}{y + 3}.$$

For the particular solution,

$$y + 3 = \sqrt{4 + \frac{2x^3}{3}}.$$

The denominator is positive wherever the solution is defined. Also, $x^2 > 0$ for $x \neq 0$, and $x^2 = 0$ at $x = 0$.

The solution is defined when

$$4 + \frac{2x^3}{3} > 0,$$

which gives

$$x > -\sqrt[3]{6}.$$

Therefore, the particular solution is increasing on

$$(-\sqrt[3]{6}, 0) \cup (0, \infty).$$